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ON FIBERING FOUR- AND FIVE-MANIFOLDS

BY
SHMUEL WEINBERGER†
Department of Mathematics, Bar-Ilan University, Ramat Gan, Israel; and Department of Mathematics, University of Chicago, Chicago, Illinois, USA

ABSTRACT
This paper studies the question of which manifolds fiber over the circle in dimensions four and five.

When is a map $f: M^n \to S^1$ homotopic to the projection of a fiber bundle? Stallings gave a complete answer for $n = 3$ [St]. For $n \geqq 6$, Browder and Levine [BL] solved the problem assuming $\pi_1 M = \mathbf{Z}$; Farrell [Fa] solved the general case. In this paper we make some observations about the missing dimensions.

First of all, the results have all been independent of the category considered. One can see directly that a smooth manifold which topologically fibers, fibers smoothly at least for $n > 5$. For $n = 5$ one can consider the product of a circle and the $E_8 + E_8$ manifold constructed by Freedman [Fr1]. Though this has a smooth structure, it is not a smooth fibration in view of the work of Donaldson [D].†† One sees that fibering 5-manifolds is closely related to existence problems for 4-manifolds. The four-dimensional case is also related to difficult s-cobordism problems. Consequently, we prefer to work topologically.

One result is that Farrell's theorem is valid for 5-manifolds whose fundamental groups do not grow too quickly [Fr2] (see also [FrQ]). Felix Hsu has proven this in his 1985 Michigan Ph.D. thesis by adapting the old proof to utilize Freedman's new low-dimensional handlebody theory. We sketch a different proof based on high dimensional "splitting" ideas that leads to a result on "Λ-splitting" four-dimensional homotopy equivalences with small fundamental groups (see [FrT]).

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†† A. Ranicki has pointed out to me that C. Kearton has given an example of a smooth knotted S^3 in S^5 that is not smoothly fibered as a consequence of [D].

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Our second result is a four-dimensional failure, or rather, two failures of different sorts. The first is a manifold which is simple-homotopy equivalent to $S^1 \times \mathbf{R}P^3$ but does not fiber since there is no submanifold homotopy equivalent to $\mathbf{R}P^3$ embedded in it in the homologically correct fashion. (There is a manifold of the right homology either by the construction or by Λ-splitting.) The proof of this uses a recent result of Casson to the effect that any homotopy three sphere has vanishing Rochlin invariant.

The second failure is due to Cappell and Shaneson [CS] and can be described as the result of glueing together the boundary components of their fake s-cobordism of a quaternionic space form to itself. There a single fiber is obviously visible, but there is no 1-parameter family of fibers. The difficulty in the proof comes in showing that no other splitting of the 4-manifold along a "fiber" yields a product.

There are no known obstructions to smoothing either example.[†] Both examples also display three dimensional aspects. The first, of course, relies on Casson's theorem on the nonexistence of certain three manifolds; the second on the nonexistence of exotic diffeomorphisms of three dimensional quaternionic space forms. In [We1] other four-dimensional "anomalies" related to three-dimensional phenomena are presented; these regard cutting and pasting, and a paucity of invariant codimension one submanifolds for diffeomorphisms of four-manifolds homotopic to the identity.

Section 1 gives our positive results and Section 2 the negative ones. These sections are independent. It is a pleasure to thank Sylvain Cappell and Julius Shaneson for their friendly conversations and to also thank Reinhardt Schultz for pointing out an error in an early version of this paper. I am also grateful to the referee for many suggestions that have improved the exposition.

1. The positive results

We prove the following:

THEOREM 1. *An n-dimensional homotopy equivalence can be topologically Λ-split, for $n \geqq 4$, provided all fundamental groups involved are "small", if and only if it can be split after crossing with $\mathbf{C}P^4 \# S^3 \times S^5 \# S^3 \times S^5$. If $n \geqq 5$, Λ-split can be replaced with split.*

The condition of "smallness" on a group is one on the growth of the number

[†] See note added in proof.

of words that can be described as a product of a given number of generators. Poly- (finite or cyclic) groups are all "small".

The proof is a version of the high dimensional methods of [B], [Wa]. The case treated there is established even for smooth 4-manifolds in [FrT].

Two comments should be made. First, Cappell [Cp] has given a very complete analysis of the high dimensional splitting problem, so Theorem 1 combines with his work to give useful low-dimensional information. Second, a direct proof of this theorem would give a new proof of (the known cases of) topological surgery, but we instead deduce Theorem 1 from the results of [Fr2] [FrQ].

We recall the definition of (Λ-) splitting. The setup is this. Given is a homotopy equivalence $f: M^n \to X$ from a manifold to a Poincaré complex and a decomposition $X = X_1 \cup_Y X_2$ where (X_i, Y) are Poincaré pairs. (It is also possible to consider the case where Y does not separate, as is relevant for deducing the Farrell fibration theorem.[†]) We say f is split or Λ-split if $f|_{f^{-1}Y}$ is a homotopy equivalence or a $\mathbf{Z}[\pi_1 Y]$ homology equivalence; f is Λ-splittable if f is homotopic to a (Λ-) split map. If $n \geqq 6$, f is splittable if and only if it is Λ-splittable. I do not know whether this is true for $n = 5$ (if $\pi_1 Y$ is small, then it is); it is not true for $n = 4$ in light of the results of Section 2.

We write down the details in the more difficult case of four-manifolds.

Proof of Theorem 1. All L-groups are L^h-groups. A key point is that surgery on three manifolds Λ-works. That is, solvable surgery problems give rise to manifolds Λ-homology equivalent to their targets. Also, one can realize the action of $L_4(\pi_1 M^3)$ if one is willing to obtain a Λ-equivalent manifold at the other end. (See [CS2] for an interpretation.) The reason is simply that all embeddings can be produced by general position, and homology calculations go exactly the same way as in [Wa].[††] Consequently, four dimensional problems with small π_1 and with boundary can be solved, but the map on the boundary will only be a Λ-equivalence.

[†] To fiber $g: M \to S^1$, consider the natural homotopy equivalence of M to the mapping torus of the monodromy of the infinite cyclic cover as a first step. Splitting along the infinite cyclic cover produces a "prefiber". Cutting along this prefiber results in a self h-cobordism which must be shown to be a product.

[††] A simple consequence is that it is now possible to compute, say, the topological concordance classes of, say, free $\mathbf{Z}_n$ actions on homology 3-spheres. Then Wall's desuspension theorem for odd order cyclic group actions remains valid, that is, using the double suspension theorem, every $\mathbf{Z}_{2k+1}$ action on S^5 is of the form $S^1 * \Sigma^3$ where $\mathbf{Z}_{2k+1}$ acts freely on S^1 and on Σ^3, and the action of Σ is well-defined up to concordance. For more information and some of the implications of this see [We2]. This example is implicit in the example of Theorem 2.

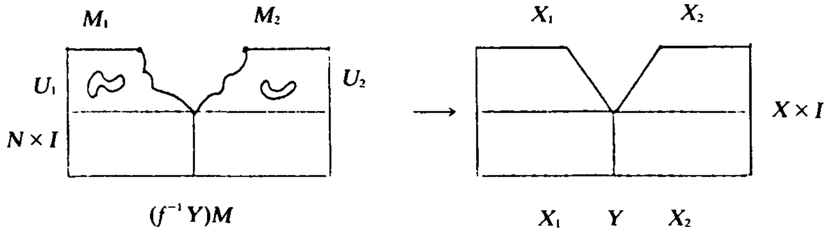

Fig. 1.

Notice that the homotopy class of f determines obstructions in $L_4(\pi_1 X_i, \pi_1 Y)$ determined by the obstruction of $f\,|_{f^{-1}(X_i, Y)}$. If f (Λ-) splits, obstructions vanish. Since $f \times 1_W$ splits, periodicity implies these obstructions vanish. Cobord these problems to solutions M_i via U_i. See Fig. 1.

Now try to cobord $M \times I \cup U_1 \cup U_2$ to a homotopy equivalence rel $M \times 0 \cup M_1 \cup M_2$. The obstruction lies in $L_5(\pi_1 X, \pi_1 Y)$. It is only well defined modulo the images of $L_5(\pi_1 X_i, \pi_1 Y)$. However, modulo those images the obstruction does vanish, again by periodicity. So act by $L_5(\pi_1 X_i, \pi_1 Y)$ on the M_i to obtain a vanishing obstruction and then surger. This produces an h-cobordism of M to a Λ-split map. A last application of periodicity guarantees (after some more modifications now by the action of a Whitehead group) that the torsion vanishes, so that, since $\pi_1 X$ is small, it is a product thus yielding a homotopy. Q.E.D.

REMARKS. The same method can be used to establish a Λ-Browder–Livesay desuspension theory for one-sided submanifolds. Julius Shaneson pointed out that the methods of the next section show that splitting also fails in this case. (Think about fake $\mathbf{R}P^4$'s.)

I do not know the status of the Λ-splitting results of this section smoothly. They are valid if $\pi_1 Y = \pi_1 X_1$ according to [FrT].

2. The negative results

Consider the Brieskorn homology sphere $\Sigma(5,7,11)$ with its natural free involution. One can check that $\mu(\Sigma) \neq 0$. Now surger circles normally generating $\pi_1(\Sigma)$ in $S^1 \times (\Sigma/\mathbf{Z}_2)$. An easy application of [Fr2] shows that this manifold has a decomposition as $W \# n(S^2 \times S^2)$, where W^4 is simple homotopy equivalent to $S^1 \times \mathbf{R}P^3$. It is easy to check that W satisfies the hypotheses (besides dimension!) of Farrell's theorem, i.e., that the infinite cyclic cover is homotopy equivalent to a finite complex and an element of the Whitehead group (which vanishes) is zero. Nonetheless:

THEOREM 2. *W does not fiber over S^1. Moreover, there is no "prefibering", i.e., no $M^3 \to W$ is equivalent to the inclusion of the homotopy fiber of $W \to S^1$.*

We show in fact that there is no three-manifold N^3 such that the infinite cyclic cover $\tilde{W} \approx N \times \mathbf{R}$. (Such a manifold would of course be a mod 2 homology sphere.) Thus none of the cyclic covers fiber.

PROOF OF THEOREM 2. We prove the result as a consequence of the following facts. A topological spin manifold is by definition just a manifold whose first two Stiefel–Whitney classes are zero.

(1) If M^3 is a mod 2 homology 3-sphere with free involution then $M/\mathbf{Z}_2$ is spin cobordant to $\mathbf{R}P^3$ by a cobordism which maps to $\mathbf{R}P^3$.

(2) The signature of the 2-fold cover of any such topological spin cobordsim is determined mod 16.

(3) The signature is just $\mu(M)$.

A straightforward calculation yields $[\mathbf{R}P^3 \colon G/\mathrm{Top}]\mathbf{Z}_2$ detected by a codimension one Arf invariant. The maps $M/\mathbf{Z}_2 \to \mathbf{R}P^3$ and id: $\mathbf{R}P^3 \to \mathbf{R}P^3$ are both homology equivalences and so are cobordant. This establishes (1). As for (2), let V_1 and V_2 be two cobordisms. Since M is a mod 2 homology sphere, and there are maps to $\mathbf{R}P^3$, $V_1 \cup_{(\mathbf{R}P^3 \cup M/\mathbf{Z}_2)} V_2$ is a topological spin 4-manifold

$$\begin{aligned} \operatorname{sign} \tilde{V}_1 - \operatorname{sign} \tilde{V}_2 &= \operatorname{sign}(\widetilde{V_1 \cup V_2}) \\ &= 2 \operatorname{sign} V_1 \cup V_2 \\ &\equiv 0 \bmod 16, \end{aligned}$$

since the signature of any topological spin manifold is divisible by 8. (3) is trivial as one can take a smooth spin cobordism and use this to compute μ.

Suppose now that $\tilde{W} = M \times \mathbf{R}$. By construction $\tilde{W}$ contains $\Sigma/\mathbf{Z}_2$ as a submanifold homology equivalent to H. Since $\Sigma/\mathbf{Z}_2$ is compact one can find a copy of M "far out" in the $\mathbf{R}$-direction that does not intersect it. These submanifolds bound a $\mathbf{Z}[\mathbf{Z}_2]$-homology h-cobordism. So (1), (2), and (3) imply $\mu(\Sigma) = \mu(\tilde{M})$. $\tilde{M}$ is, of course, a homotopy 3-sphere and this contradicts Casson's theorem. Q.E.D.

Now we consider X obtained by glueing the boundary components of the Cappell–Shaneson fake s-cobordism [CS1] together.

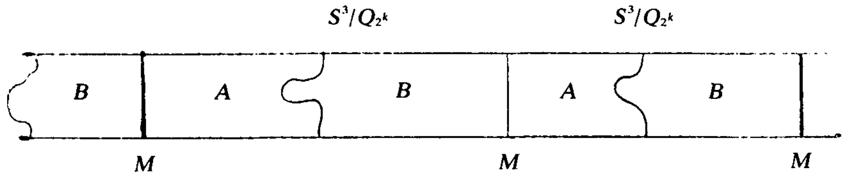

Fig. 2.

THEOREM 3. *X prefibers but does not fiber.*

In fact, the infinite cyclic cover $\tilde{X}$ is $(S^3/Q_{2^k})\times\mathbf{R}$. (The two-fold cover is $S^1\times(S^3/Q_{2^k})$.)

PROOF OF THEOREM 3. A prefiber is given by the obvious S^3/Q_{2^k} in X. The proof in [CS1] shows that the result of cutting X along any S^3/Q_{2^k} gives a nontrivial s-cobordism, not a product. The issue is the possible existence of some other M^3 prefiber for which this is not true. (After all, we do not understand the diffeomorphisms of such a hypothetical M.) Taking an odd-fold cover if necessary, one can assume that M and S^3/Q_{2^k} are disjoint. Consider Fig. 2. One knows that $A\cup B=M\times I$ but that $B\cup A\neq S^3/Q_{2^k}\times I$. Thus $B\cup A\cup B=B$. However, h-cobordisms with small fundamental group can be seen to be invertible using [Fr2] (e.g., B is, since $(B\cup A)\times I=M\times D^2$). So $B\cup A$ is in fact trivial, giving a contradiction. Q.E.D.

Note added in proof. Cappell and Shaneson (Bulletin of the American Mathematical Society, 1987) have now given smooth counterexamples to the s-cobordism theorem in dimension four. These lead to smooth four-dimensional nonfibering manifolds as in Theorem 3.

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